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A STUDY OF NONLINEAR LONGITUDINAL DYNAMIC STABILITY

by Murray Tobak and Walter E. Pearson

Ames Research Center
Moffett Field, Calif.

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SUMMARY

Principles from functional analysis are used as the framework for a reformulation of the notions of aerodynamic indicial functions and superposition. An integral form for the aerodynamic pitching-moment coefficient is derived which is free of dependence on a linearity assumption. Results are applied to the study of nonlinear longitudinal dynamic stability. Implications of the results are discussed, especially in regard to the meaning of experimental determinations of nonlinear aerodynamic damping derivatives.

Auth.

INTRODUCTION

The history of mathematical studies of the dynamic stability of aircraft begins essentially with Bryan's "Stability in Aviation" (ref. 1). It is a tribute to its penetration that this analysis, published at the very beginning of heavier-than-air flight itself, has remained the foundation for practically all subsequent studies of the subject. The assumption central to Bryan's formulation is that the aerodynamic forces and moments developed at a given instant depend only on the instantaneous values of the variables which determine forces and moments in a steady flow. When, in addition, a linear dependence of the forces and moments on these variables is assumed, the equations governing the motions of aircraft reduce to a set of linear ordinary differential equations having constant coefficients. With the form of the equations thus established, the study of stability becomes synonymous with study of the coefficients. The nature and determination of these coefficients, the "stability derivatives," has been the central concern of experimenters and analysts alike through the ensuing years.

Later investigations into the transient behavior of the aerodynamic forces and moments in response to sudden changes in the flow around the aircraft led researchers to recognize that the forces and moments at an instant were dependent not only on the instantaneous values of the flow variables but also on their past values (cf. ref. 2 for a comprehensive summary and bibliography). The concept of transient aerodynamic force and moment responses to step changes in the flow variables, that is, of "indicial functions," coupled with the notion of superposition, led to a new formulation of the equations of motion (ref. 3). In this formulation, which is exact in principle within the assumption of linearity, the equations of motion take the form of integro-differential equations. As these generally are more difficult to treat than the original ordinary differential equations, the new formulation has not been widely adopted in practice. Its chief contribution has been to provide a framework within which the deficiencies of the original formulation could be appreciated and perhaps removed. This has been the case in the study of aircraft stability, in virtue of the very low frequencies characteristic of aircraft motions. Reduction of the complete equations to equations correct to the first order in frequency restores the form of the original equations, the latter now properly including terms accounting for the past within the order of the approximation. In effect, the more exact formulation provided the basis on which this improvement could be made, and in addition, a means of clarifying the nature of the added terms, terms owing their existence to time-dependent phenomena (ref. 4).

While the original equations of motion, amended as described, heretofore have been capable of reliably depicting the motions of aircraft, there is considerable evidence that they now fail to do so in a consistent way. The cause of this failure has a clear connection with the remaining assumption basic to the original formulation, namely, that of linearity. In response to the demand for flight at extreme altitudes and speeds, the aircraft's evolution has tended to merge with that of the projectile; generally having only marginal stability and operating under extreme flight conditions, such vehicles often undergo motions of large amplitude. Over the ranges of these large-amplitude motions, it is no longer an adequate approximation to represent the aerodynamic forces and moments as linear functions of the flow variables. It has become apparent, then, that a reformulation of the aerodynamic forces and moments is again in order, now free of the assumption of linearity. This reformulation should be expected to be a generalization of the exact linear formulation mentioned previously. Hence, it is reasonable to anticipate that a new formulation of the aerodynamic forces and moments should introduce integral forms which are essentially generalizations of the notion of superposition. The term integral forms is suggestive of the mathematical theory of functionals (ref. 5), and, indeed, this theory is found to be of sufficient breadth to provide the framework for the desired reformulation.

In this report, fundamental principles of functional analysis are used to construct a framework within which the indicial function can be reformulated as a nonlinear functional; the result is a new definition for the indicial function which does not depend on a linearity assumption. This definition leads naturally to the derivation of integral forms for the aerodynamic forces and moments which are the anticipated generalizations of the notion of superposition.

Although in principle the analysis can be extended to more general problems, for simplicity it is restricted here to the problem of longitudinal motion alone. Even for this restricted problem, the results no doubt are too complex to be of immediate practicability; however, just as was the case for the exact linear formulation, it is found possible to reduce the results to simpler more practicable forms in virtue of the low frequencies characteristic of aircraft motions. Implications of the results are discussed, especially in regard to the meaning of experimental determinations of aerodynamic damping derivatives.

NOTATION

C_L	lift coefficient, $\frac{\text{lift}}{q_o S}$
C_m	pitching-moment coefficient, $\frac{\text{pitching moment}}{q_o S l}$
C_N	normal-force coefficient, $\frac{\text{normal force}}{q_o S}$
$C_{m_\alpha}[\alpha(\xi),q(\xi);t,\tau]$	indicial pitching-moment response measured at t per unit step change in α occurring at τ, with q held fixed at $q(\tau)$
$C_{m_q}[\alpha(\xi),q(\xi);t,\tau]$	indicial pitching-moment response measured at t per unit step change in q occurring at τ, with α held fixed at $\alpha(\tau)$
e	base of natural logarithms
$G[u(\xi),v(\xi)]$	functional notation: value at $\xi = t$ of a function $F(t)$ which is dependent on all the values taken by the two argument functions $u(\xi),v(\xi)$ over the interval $0 \leq \xi \leq t$
I	moment of inertia about axis of pitch
i	$\sqrt{-1}$
k	reduced frequency, $\frac{\omega l}{V_o}$
l	characteristic length
q	dimensionless pitching-velocity parameter, $\frac{\dot{\theta} l}{V_o}$
q_o	dynamic pressure, $\frac{1}{2} \rho_o V_o^2$
S	characteristic area
s	distance traveled along the flight path
t	time
t_a	time required following an instantaneous change in angle of attack or pitching velocity for the indicial pitching moment to attain steady state
V_o	flight speed (fig. 1)
x,y,z	Cartesian coordinate system with origin at aircraft center of gravity (fig. 1)
α	angle of attack (fig. 1)
α_m	mean angle of attack (fig. 12)
α_o	oscillation amplitude (fig. 12)
θ	angle of pitch (fig. 1)

ξ running variable in time

ρ_o atmospheric density

σ_a number of characteristic lengths traveled in time t_a, $\frac{V_o t_a}{l}$

τ value of time ξ at which step change in α or q occurs

φ number of characteristic lengths traveled in time t, $\frac{V_o t}{l}$

ω circular frequency

α', α'' $\frac{d\alpha}{d\varphi}, \frac{d^2\alpha}{d\varphi^2}$

$\dot{\alpha}$ $\frac{d\alpha}{dt}$

When α, $\dot{\alpha}$, and q are used as subscripts with a pitching-moment coefficient, a dimensionless partial derivative is indicated; thus

$$C_{m_\alpha} = \frac{\partial C_m}{\partial \alpha} \quad , \quad C_{m_{\dot{\alpha}}} = \frac{\partial C_m}{\partial(\dot{\alpha} l / V_o)} \quad , \quad C_{m_q} = \frac{\partial C_m}{\partial q}$$

Conventional use of the mathematical symbols O and ϵ is made. The notation $x \epsilon A$ is shorthand for "x is an element of the set A;" the notation $y = O(x)$ is shorthand for "y is of the order of x." Open and closed intervals are indicated by the symbols ( ) and [ ], respectively. The symbol ( ] means that the interval is open on the left and closed on the right.

GENERAL CONSIDERATIONS

Definition of Coordinates

In the study to follow, the aircraft is considered to be in level steady flight prior to time zero; at time zero it begins a strictly longitudinal maneuver in which the altitude changes are sufficiently small that the atmospheric density along the flight path remains essentially constant. It is further specified that the aircraft's velocity along the flight path remain constant. Hence, dynamic pressure and Mach number, as measured along the flight path, also remain fixed throughout the motion.

It is convenient to define an orthogonal coordinate system which follows the path of the aircraft's center of gravity, and, to this end, the origin of coordinates is attached to the center of gravity. With the analysis restricted to longitudinal motions only, the origin never departs from a vertical plane; its movement in the plane is defined by the velocity V_o, a vector of fixed magnitude. As shown in figure 1, the positive branch of the x axis is alined with the direction of V_O; the y axis, perpendicular to the aircraft's vertical plane of symmetry, is coincident with the axis of rotation; the z axis lies in the vertical plane, positive downward.

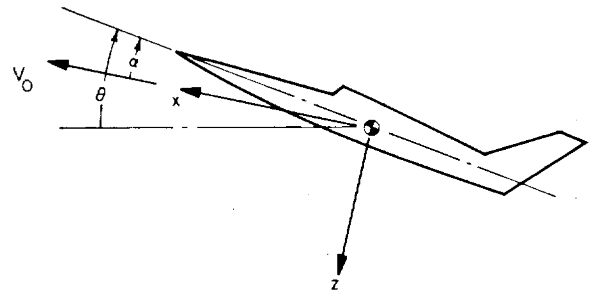
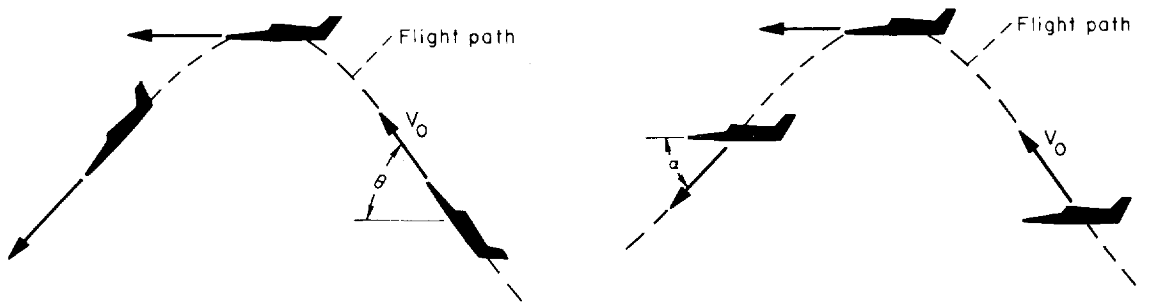


Figure 1.- Definition of coordinates.

Figure 1 also introduces the angles α and θ, which serve to define the aircraft's maneuver in the vertical plane. Angle of attack α is defined as the angle between the chord plane of the main lifting surface and the xy plane; it is shown as positive in figure 1. Angle of pitch θ is the angle between the same chord plane and the horizontal plane (an arbitrary reference). With V_O constant, the translatory and angular motions of the aircraft are defined with respect to any system of coordinates when α and θ are known. In figure 2, two motions having the same flight path are shown in order to



(a) Angle of pitch = θ, angle of attack = 0.

(b) Angle of pitch = 0, angle of attack = α.

Figure 2.- Maneuvers corresponding to purely (a) angle of pitch and (b) angle-of-attack variations.

illustrate the difference between a maneuver which involves a constant angle of attack with a varying angle of pitch and one which involves a constant angle of pitch with a varying angle of attack.

Now consider a maneuver involving simultaneous arbitrary variations of α and θ. In response to the maneuver, aerodynamic forces and moments are developed which depend on flow conditions at the surface of the aircraft. This dependence is characteristically determined by the nature of the flow <u>velocities</u> which impinge on the surface. Consider in particular the instantaneous component of flow velocity normal to the chord plane of the main lifting surface. As shown in figure 3, it consists of two

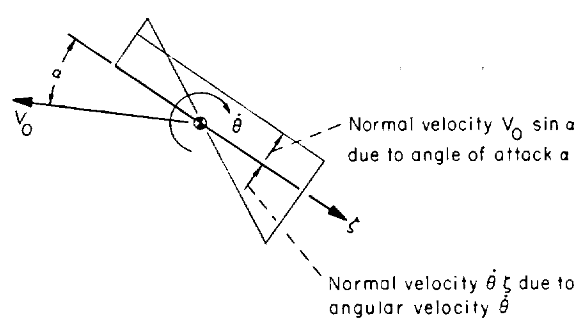


Figure 3.- Flow conditions normal to the wing chord plane.

parts, a uniform distribution of velocity $V_o \sin \alpha$ due to the instantaneous angle of attack α, and a distribution of velocity $\dot{\theta}\zeta$ linearly dependent on chordwise distance from the axis of rotation. It will be noted that the latter component is due to the angular velocity $\dot{\theta}$, and hence it is more appropriate to designate $\dot{\theta}$ rather than θ itself as one of the flow variables on which the forces and moments depend. For this purpose, the dimensionless parameter q is introduced, where $q = \dot{\theta}l/V_o$, l being a characteristic length. Then, with V_o constant, it may be said that the aerodynamic forces and moments depend on the variables α and q. At a given instant, however, they depend not only on the instantaneous values of α and q but on all the past values of these variables. This is demonstrated in the following section.

Linear Aerodynamic Pitching-Moment Response

It is intended to demonstrate that the instantaneous values of aerodynamic force and moment are indeed dependent on the whole past of α and q. As this can be clearly shown within the framework of the exact linear formulation, a summary description of this formulation will serve both as demonstration and as brief review of the notion of aerodynamic indicial functions and of superposition.

Let the aircraft begin a maneuver at time zero involving arbitrary variations in α and q, and consider the aerodynamic pitching-moment response to the angle-of-attack variation.[1] As shown in figure 4, the angle-of-attack variation may be broken into a large number of small step changes. In response to a step change in α at time τ, there is an incremental change in pitching moment, ΔC_m; it is measured at a fixed time t subsequent to τ. The assumption of linearity is now invoked, which, in the context of this report, has the following meaning: ΔC_m is said to be independent of $\alpha(\tau)$, $q(\tau)$, and the past values of these variables. This enables one to write ΔC_m in the form $\frac{\Delta C_m}{\Delta \alpha}(t - \tau)\Delta\alpha$. The form implies that $\Delta C_m/\Delta\alpha$ is derivable from a linear differential equation. That $\Delta C_m/\Delta\alpha$ depends only on elapsed time $t - \tau$, rather than on t and τ separately, implies in addition that the coefficients of the differential equation are independent of time. The following distinction should be

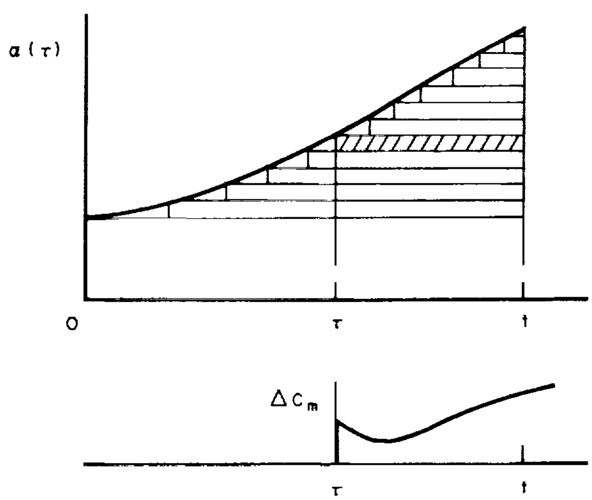


Figure 4.- Superposition of incremental responses.

[1]Hereafter in this report, for brevity, attention will be focused solely on the aerodynamic pitching moment. All that is said, however, will hold as well for the aerodynamic lift or normal force merely on the substitution of C_L or C_N for C_m.

carefully noted: The significance of the linearity assumption does not lie in the fact that ΔC_m is said to be linearly dependent on $\Delta\alpha$, for this can always be asserted whenever $\Delta\alpha$ is so small that terms of $O((\Delta\alpha)^2)$ can be neglected. The significance of the assumption lies in the fact that the ratio $\Delta C_m/\Delta\alpha$ is said to be independent of α (and q). Thus, no matter how large the value of α at the origin of the step, the response function $\Delta C_m/\Delta\alpha$ is said to be the <u>same</u> function, dependent only on elapsed time $t - \tau$. The limit of this function as $\Delta\alpha$ approaches zero

$$\lim_{\Delta\alpha\to 0} \frac{\Delta C_m}{\Delta\alpha}(t - \tau) = C_{m_\alpha}(t - \tau) \tag{1}$$

is called the indicial pitching-moment response per unit step change in α. Every step beginning at a value of τ less than t has a corresponding incremental indicial response which contributes to the pitching moment at time t. The summation of these incremental responses to the steps which occur over the range of τ from zero to t then gives the pitching moment at time t due to the variation in α. As the indicial response depends only on the time difference $t - \tau$, in the limit the summation takes the form of the familiar convolution integral. The contribution to the total pitching moment due to the variation in q is obtained similarly, on introduction of the function $C_{m_q}(t - \tau)$, the indicial pitching-moment response per unit step change in q. The sum of the two contributions and the initial value of C_m then gives the total pitching moment at time t

$$C_m(t) = C_m(0) + \int_0^t C_{m_\alpha}(t - \tau) \frac{d}{d\tau}\alpha(\tau)\,d\tau + \int_0^t C_{m_q}(t - \tau) \frac{d}{d\tau} q(\tau)\,d\tau \tag{2}$$

For constant V_o and within the assumption of linearity, equation (2) is exact. As all values of α and q figure within the limits of the integrals, the equation confirms the assertion that C_m at time t depends on the whole past of α and q.

It is desired to derive an integral form for $C_m(t)$, analogous to equation (2), free of the assumption of linearity. This form must of course include the linear form as a special case, and therefore those features of the linear form which are independent of the assumption of linearity must be retained. Accordingly, the problem can be stated in this general way: For a motion at constant V_o beginning at time zero and involving arbitrary continuous variations in α and q, find an integral form for C_m at time t which is functionally dependent on α and q and on all the values taken by α and q within the time interval zero to t. Stated in this way, the description of $C_m(t)$ corresponds mathematically to Volterra's description of a <u>functional</u> (ref. 5). If the notation of reference 5 is adopted, the assertion that $C_m(t)$ is a functional is indicated thus,

$$C_m(t) = G\left[\alpha(\overset{t}{\underset{o}{\xi}}), q(\overset{t}{\underset{o}{\xi}})\right] \tag{3}$$

where it is understood that ξ is a running variable in time, ranging over the interval zero to t. The attractive feature of the theory of functional analysis is that, in its most general form, it does not rest on an assumption of linearity. Hence, an appeal to the general theory may open the way to the desired formulation. As the theory is relatively unfamiliar to aerodynamicists, those elements of it relevant to the present problem are reviewed briefly in the following section.

FUNDAMENTALS OF FUNCTIONAL ANALYSIS

Basic Concepts and Notation

The mathematical concept of a functional is a generalization of the familiar notion of a function. A brief review of this notion will provide a means of introducing functionals.

When y is a function of a variable x in an interval $[a,b]$, the usual notation is

$$y = f(x) \quad , \qquad a \leq x \leq b$$

It is understood that for each x in the interval, the function, f, assigns a number, $f(x)$, to y. Thus, a function assigns a number to each member of a collection of points; this collection need not be a closed interval, but can be an arbitrary set. Whatever its nature, the collection of points for which the function is defined is called its domain.

A functional also assigns a number to each member of a collection, but the domain of a functional is a collection of functions instead of a set of points. That is, given a collection of functions of x which are all defined in some interval $[a,b]$, the functional assigns a number to each member of the collection. If a functional F assigns a number z to a function $g(x)$ from the collection, it is denoted by

$$z = F\left[g\left(\overset{b}{\underset{a}{x}}\right) \right]$$

The end points $\overset{b}{\underset{a}{}}$ are frequently omitted from this notation because the interval over which the functions of the collection are being considered is usually defined *a priori*. The common notation is

$$z = F[g(x)]$$

where it is understood that the number z depends on all those ordinates $g(x)$ which correspond to abscissas, x, contained in an interval previously defined. This tacit understanding tends to obscure the fact that a functional may be looked upon as a function of "several" variables. It is, in fact, instructive to consider a functional as a function with an entire continuum of independent variables.

The definite integral is a familiar example of a functional. As one example of a collection of functions, consider those functions which are continuous in the interval $0 \leq x \leq 1$. Let the collection be called C_{01} and let $g(x)$ be any member of this collection. Then the definite integral

$$\int_0^1 g(x)dx$$

exists and is a fixed number. It is therefore possible to define the functional F with domain C_{01} by the relation

$$F[g(x)] = \int_0^1 g(x)dx$$

The functional F assigns a number to every $g(x)$ in the collection C_{01}. Since this number is the area under the graph of $g(x)$, it is quite clear that the number assigned by F depends on the continuum of values taken on by $g(x)$ in the interval. This particular functional is also an example of a linear functional which will now be defined.

The domain of a functional is usually closed under the operations of addition of functions and multiplication by constants. That is, when $\varphi(x)$ and $\psi(x)$ are functions in the domain, then so is the function

$$\sigma(x) = c\varphi(x) + k\psi(x)$$

where c and k are any constants. The domain C_{01} has this closure property. If $\varphi(x)$ and $\psi(x)$ are functions in C_{01}, they are continuous in the interval, $[0,1]$, and it follows immediately that the function $\{c\varphi(x) + k\psi(x)\}$ is also continuous in this interval. Hence, it is a function in the collection C_{01}. (It should be understood that this continuity condition applies merely to the collection C_{01} being considered; in general, continuity is not a requisite for the functions of a domain.)

A functional, G, is said to be a linear functional if

$$G[c\varphi(x) + k\psi(x)] = cG[\varphi(x)] + kG[\psi(x)]$$

for every pair of functions $\varphi(x)$ and $\psi(x)$ in its domain, and arbitrary constants c and k.

The elementary properties of an integral make the functional

$$F[g(x)] = \int_0^1 g(x)dx$$

a linear functional. This can be seen from its definition, $F[g(x)] = \int_0^1 g(x)dx$ so that

$$\begin{aligned} F[c\varphi(x) + k\psi(x)] &= \int_0^1 \{c\varphi(x) + k\psi(x)\}dx \\ &= c\int_0^1 \varphi(x)dx + k\int_0^1 \psi(x)dx \\ &= cF[\varphi(x)] + kF[\psi(x)] \end{aligned}$$

Derivative of a Functional

The analogy between the derivative of a functional and the ordinary derivative of a function is quite direct, and an examination of the ordinary derivative reveals much of the nature of the derivative of a functional.

Initially, the derivative of a function $f(x)$ is defined at an arbitrary but fixed point x_o of its domain. An infinitesimal increment Δx is added to x_o and the two numbers $f(x_o)$ and $f(x_o + \Delta x)$ are compared. The increment Δf corresponding to Δx is

$$\Delta f = f(x_o + \Delta x) - f(x_o)$$

and the derivative of $f(x)$ at x_o is defined by

$$f'(x_o) = \lim_{\Delta x \to o} \frac{\Delta f}{\Delta x}$$

If the limit $f'(x_o)$ exists for each point x_o in some part of the domain of $f(x)$, the derivative is itself a function, $f'(x)$.

Analogously, the derivative of a functional $F[g(x)]$ is first defined at a "point," $g_o(x)$, of its domain. An incremental function $\epsilon\varphi(x)$ is added to $g_o(x)$, and the corresponding increment ΔF in the functional is given by

$$\Delta F = F[g_o(x) + \epsilon\varphi(x)] - F[g_o(x)]$$

As expected, the derivative of the functional is then defined by means of a limit, but the nature of this limit must be carefully stated.

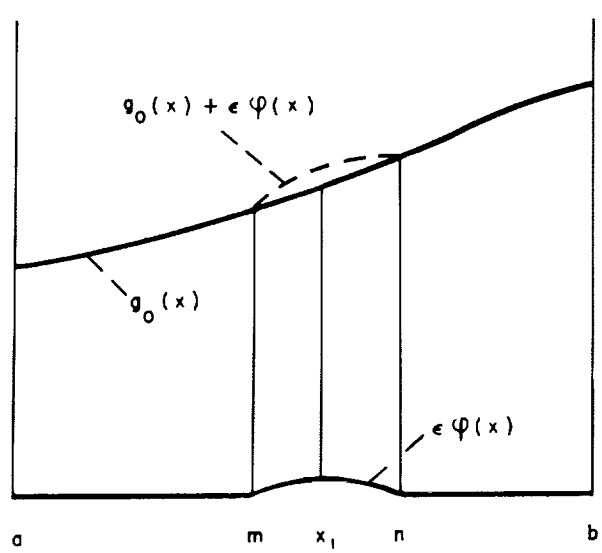


Figure 5.- Illustration of $g_o(x) + \epsilon\varphi(x)$.

A point x_1 is chosen inside the interval $[a,b]$ where F operates on the functions of its domain. The arbitrary, but fixed, function $g_o(x)$ is given the increment $\epsilon\varphi(x)$ in such a way that $\varphi(x) > 0$ in a subinterval (m,n) which contains the point x_1, but such that $\varphi(x) \equiv 0$ outside (m,n), as is shown in figure 5. The norm $\|\epsilon\varphi(x)\|$ of the incremental function and the number μ will be defined here as

$$\|\epsilon\varphi(x)\| = \int_a^b |\epsilon\varphi(x)|\,dx = |\epsilon| \int_m^n \varphi(x)dx$$

$$\mu = \max_{m \leq x \leq n} \{|\epsilon\varphi(x)|\}$$

The symbol $\|\epsilon\varphi(x)\|_{x_1} \to 0$ means that both $\mu \to 0$ and $(n - m) \to 0$ simultaneously and, moreover, that $(n - m) \to 0$ in such a manner that the point $x = x_1$ always lies within the interval (m,n).

If $\lim_{\|\epsilon\varphi(x)\|_{x_1}\to 0} \dfrac{\Delta F}{\|\epsilon\varphi(x)\|}$ exists, this limit is defined to be the derivative of F with respect to $g_o(x)$ at the point $x = x_1$. The notation adopted to express this is

$$\left(\frac{dF}{dg_o}\right)_{x=x_1} = F'[g_o(x);x_1] = \lim_{\|\epsilon\varphi(x)\|_{x_1}\to 0} \frac{F[g_o(x) + \epsilon\varphi(x)] - F[g_o(x)]}{\|\epsilon\varphi(x)\|}$$

It is clear that the limit depends on the function $g_o(x)$ as well as on the point x_1. Therefore, the derivative is both a functional and a function. When the point x_1 is fixed and the derivative $F'[g(x);x_1]$ exists for each $g(x)$ in the domain of F, $F'[g(x);x_1]$ is a functional with the same domain as F. When a particular function $g_o(x)$ is chosen and the derivative exists for each $a < x_1 < b$, F' is a function of x_1 in the interval (a,b).

It is interesting to note that if $F[g(x)]$ is again defined to be a definite integral

$$F[g(x)] = \int_a^b g(x)\,dx$$

then

$$F'[g(x);x_1] = \mp 1 \quad , \qquad \epsilon \gtrless 0$$

for every $x_1 \in [a,b]$ and every $g(x)$ in the domain of F.

Derivatives of a Functional of Two Arguments

A functional may have more than one argument, just as a function may have more than one argument. When a functional has two arguments, its domain consists of ordered pairs of functions, as is indicated by the notation

$$z = F\left[g\left(\overset{b}{\underset{a}{x}}\right), h\left(\overset{d}{\underset{c}{x}}\right)\right]$$

All the possible functions for the first member of the pair are defined in an interval $[a,b]$ and the functions for the second member are defined in an interval $[c,d]$. It may happen that $[a,b] = [c,d]$. Given any function from the collection of first members <u>and</u> any function from the collection of second members, the functional F assigns a number z.

The integro-exponential function of order n, $E_n(t)$, provides a simple example of a functional of two arguments. The function $E_n(t)$ is defined for positive values of t by

$$E_n(t) = \int_1^\infty x^{-n} e^{-tx}\,dx$$

That this is a functional of two arguments becomes evident if the set

$$\{x^{-n}\} \quad , \qquad n = 1, 2, 3, \ldots \quad ; \qquad 1 < x < \infty$$

is taken for first member functions $g(x)$, and the set

$$\{e^{-tx}\} \quad , \qquad t > 0 \quad ; \qquad 1 < x < \infty$$

is taken for second member functions $h(x)$. Then every choice of n and t determines the ordered pair of functions

$$[g(x),h(x)] = [x^{-n},e^{-tx}]$$

Defining the functional as

$$F\left[g\left(\overset{\infty}{\underset{1}{x}}\right),h\left(\overset{\infty}{\underset{1}{x}}\right)\right] = \int_1^\infty g(x)h(x)\,dx$$

one obtains

$$F\left[g\left(\overset{\infty}{\underset{1}{x}}\right),h\left(\overset{\infty}{\underset{1}{x}}\right)\right] = \int_1^\infty x^{-n}e^{-tx}\,dx = E_n(t)$$

In this example, $F[g(x),h(x)] = F[h(x),g(x)]$ but this condition is not required of a functional of two arguments. Also in this example, the intervals $[a,b]$ and $[c,d]$ coincide, but this is not, in general, a requirement. Usually, the notation for a functional of two arguments is shortened by the omission of the end points.

When $h(x)$ is a fixed function, say $h(x) = h_o(x)$, the functional F assigns a number to each function which may be chosen as a first member for the ordered pair. Hence $F[g(x),h_o(x)]$ is a functional with one argument. This restriction of $F[g(x),h(x)]$ to the pairs $[g(x),h_o(x)]$ is indicated by the notation

$$F[g(x),h(x)]_{h=h_o}$$

When it is indicated that $h(x)$ is a fixed function but the function is not explicitly named, the symbol $F[g(x),h(x)]_h$ is used.

Since $F[g(x),h(x)]_h$ is a functional with one argument, its derivative is defined in precisely the same manner as the derivative of $F[g(x)]$.

$$F'[g_o(x),h(x);x_1]_h = \lim_{\|\epsilon\varphi(x)\|_{x_1}\to o} \frac{F[g_o(x)+\epsilon\varphi(x),h(x)]_h - F[g_o(x),h(x)]_h}{\|\epsilon\varphi(x)\|}$$

It is understood that x_1 is a point in the interval $[a,b]$ where first member functions are considered. In keeping with the notation for a partial derivative of a function, the symbol $F'[g_o(x),h(x);x_1]_h$ is replaced by

$$F_g[g_o(x),h(x);x_1] = F'[g_o(x),h(x);x_1]_h$$

Similarly, the functional $F[g(x),h(x)]_g$ is a functional with one argument and its derivative is defined.

$$F_h[g(x),h_o(x);x_2] = \lim_{||\epsilon\psi(x)||_{x_2}\to o} \frac{F[g(x),h_o(x) + \epsilon\psi(x)]_g - F[g(x),h_o(x)]_g}{||\epsilon\psi(x)||}$$

Here, x_2 is a point in the interval $[c,d]$ where the second member functions are considered.

The derivatives $F_g[g_o(x),h(x);x_1]$ and $F_h[g(x),h_o(x);x_2]$ are the analogs of the partial derivatives of a function $f(u,v)$ of two variables. The total differential of $f(u,v)$ is given by

$$df = \frac{\partial f}{\partial u}\, du + \frac{\partial f}{\partial v}\, dv$$

whereas the actual increment Δf, which corresponds to increments $\Delta u = du$ and $\Delta v = dv$, is

$$\Delta f = f(u + \Delta u,\ v + \Delta v) - f(u,v)$$

It is shown by methods of elementary calculus that df differs from Δf by infinitesimals of higher order than Δu and Δv. In the next section, a formula for the total differential dz of a functional with two arguments will be developed. It will be shown that dz differs from the actual increment in the functional Δz by infinitesimals of higher order than ϵ when the incremental functions are $\epsilon\varphi(x)$ and $\epsilon\psi(x)$.

Differential of a Functional With One or Two Arguments

A functional $z = G\left[\overset{b}{\underset{a}{u(\xi)}}\right]$ with one argument will now be considered. It is assumed that G has a derivative at every point of the interval (a,b). Let $u(\xi)$ be given an increment $du(\xi) = \epsilon\varphi(\xi)$, where now $\epsilon\varphi(\xi)$ may be allowed to be different from zero over the <u>entire</u> interval (a,b). There results an increment Δz defined by

$$\Delta z = G[u(\xi) + \epsilon\varphi(\xi)] - G[u(\xi)]$$

An expression for the differential

$$dz \approx \Delta z$$

will now be developed with the following two properties:

(1) The differential dz will differ from Δz by infinitesimals of higher order than ϵ.

(2) The differential dz will be a linear functional of the incremental function $du(\xi) = \epsilon\varphi(\xi)$.

That it is possible to fulfill these specifications is a consequence of the assumption that the functional G has a derivative at every point.

The incremental function $du(\xi) = \epsilon\varphi(\xi)$ can be written as the sum of functions $\varphi_i(\xi)$ $(i = 1, 2, \ldots n)$ where each $\varphi_i(\xi)$ vanishes outside a small interval. In order to construct the functions $\varphi_i(\xi)$, the interval (a,b) is subdivided into n subintervals by the $(n + 1)$ points $a = c_0, c_1, c_2, \ldots, c_{n-1}, c_n = b$. Then the functions $\varphi_i(\xi)$ are defined by

$$\begin{aligned} \varphi_i(\xi) &= \varphi(\xi), \qquad && \xi \in (c_{i-1}, c_i] \\ &= 0, && \xi \notin (c_{i-1}, c_i] \end{aligned}$$

Then, by definition,

$$\varphi(\xi) = \sum_{i=1}^{n} \varphi_i(\xi)$$

at every point in $[a,b]$ except $\xi = a$, where $\varphi_1(\xi)$ is not defined. This anomaly will have no effect on the results to be obtained because it will not affect $\|\varphi_1(\xi)\|$.

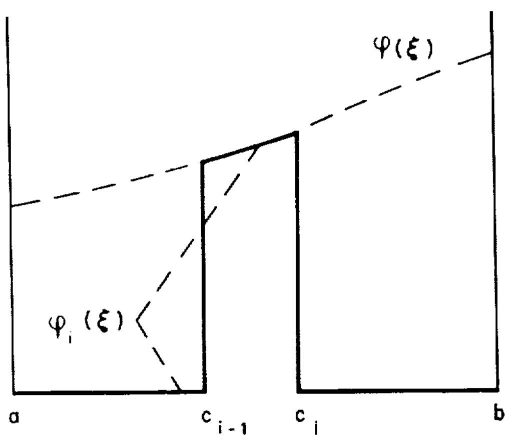


Figure 6.- The segment $\varphi_i(\xi)$.

Each of the functions $\varphi_i(\xi)$ is zero everywhere except in an interval (c_{i-1}, c_i) which can be made arbitrarily small by the insertion of more points to subdivide the interval $[a,b]$ into more subintervals. Inside the subinterval (c_{i-1}, c_i), $\varphi_i(\xi)$ can be thought of as a segment of the curve $\varphi(\xi)$, as shown in figure 6.

If the function $\epsilon\varphi_i(\xi)$ is thought of as an incremental function by itself, there corresponds an increment $\Delta_i z$:

$$\Delta_i z = G[u(\xi) + \epsilon\varphi_i(\xi)] - G[u(\xi)]$$

It is possible to estimate the magnitude of $\Delta_i z$ since the functional G is assumed to possess a derivative. By definition of G',

$$G'[u(\xi); x_1] = \lim_{\substack{\epsilon \to 0 \\ \Delta_i x \to 0}} \frac{\Delta_i z}{\|\epsilon\varphi_i(\xi)\|}$$

when $\Delta_i x = (c_i - c_{i-1}) \to 0$ in such a way that $\Delta_i x$ always contains the point x_1. Consequently, when both $|\epsilon|$ and $(c_i - c_{i-1})$ are small, it is possible to write

$$\Delta_i z = \|\epsilon\varphi_i(\xi)\| \left\{G'[u(\xi); x_1] + \eta(x_1)\right\}$$

where

$$\lim_{\substack{\epsilon \to 0 \\ \Delta_i x \to 0}} \eta(x_1) = 0$$

The mean value theorem for integrals can be used to obtain

$$\| \epsilon\varphi_i(\xi) \| = |\epsilon| \int_{c_{i-1}}^{c_i} \varphi_i(x)dx = |\epsilon| \varphi_i(x_2)\Delta_i x$$

where x_2 is a point interior to the interval (c_{i-1}, c_i). Substitution of the expression on the right for $\|\epsilon\varphi_i(\xi)\|$ gives, for a positive ϵ,

$$\Delta_i z = \epsilon G'[u(\xi);x_1]\varphi_i(x_2)\Delta_i x + \epsilon\eta(x_1) \int_{c_{i-1}}^{c_i} \varphi(x)dx$$

where both x_1 and x_2 are points interior to the interval (c_{i-1}, c_i) and where

$$\lim_{\substack{\epsilon \to 0 \\ \Delta_i x \to 0}} \eta(x_1) = 0$$

As might be anticipated, this expression for the increment $\Delta_i z$ corresponding to the incremental function $\epsilon\varphi_i(\xi)$ will be used to find the increment Δz which corresponds to the function

$$\epsilon\varphi(\xi) = \epsilon \sum_1^n \varphi_i(\xi)$$

When $G[u(\xi)]$ is a linear functional, the expressions for both $\Delta_i z$ and Δz are relatively simple, for when G is linear,

$$\begin{aligned} \Delta_i z &= G[u(\xi) + \epsilon\varphi_i(\xi)] - G[u(\xi)] \\ &= G[u(\xi)] + \epsilon G[\varphi_i(\xi)] - G[u(\xi)] \\ &= \epsilon G[\varphi_i(\xi)] \end{aligned}$$

and likewise

$$\begin{aligned} \Delta z &= G[u(\xi) + \epsilon\varphi(\xi)] - G[u(\xi)] \\ &= \epsilon G[\varphi(\xi)] = \epsilon G[\sum_1^n \varphi_i(\xi)] \\ &= \epsilon \sum_1^n G[\varphi_i(\xi)] \end{aligned}$$

This shows that when the functional G is linear,

$$\Delta z = \sum_{i=1}^n \Delta_i z$$

However, it will not be assumed that the functional G is linear, but instead a less restrictive assumption will be made. It is assumed that

$$\Delta z = \lim_{\|\Delta_i x\| \to 0} \sum_i \Delta_i z$$

where $\|\Delta_i x\|$ is the largest of the subintervals $\Delta_i x$. This assumption implies the relation

$$\Delta z = \lim_{\|\Delta_i x\| \to 0} \left\{ \sum_i \epsilon G'[u(\xi);x_{i_1}]\varphi_i(x_{i_2})\Delta_i x + \sum_i \epsilon\eta(x_{i_1}) \int_{c_{i-1}}^{c_i} \varphi(x)\,dx \right\}$$

where x_{i_1} and x_{i_2} are points interior to the interval (c_{i-1}, c_i). When both sides of this equation are divided by ϵ and the limit of the first sum on the right is taken, there results

$$\frac{\Delta z}{\epsilon} = \int_a^b G'[u(\xi);x]\varphi(x)\,dx + \lim_{\|\Delta_i x\| \to 0} \sum_i \eta(x_{i_1}) \int_{c_{i-1}}^{c_i} \varphi(x)\,dx$$

Since for each i,

$$\lim_{\substack{\epsilon \to 0 \\ \|\Delta_i x\| \to 0}} \eta(x_{i_1}) = \lim_{\substack{\epsilon \to 0 \\ \|\Delta_i x\| \to 0}} \int_{c_{i-1}}^{c_i} \varphi(x)\,dx = 0$$

it follows that

$$\lim_{\substack{\epsilon \to 0 \\ \|\Delta_i x\| \to 0}} \sum_i \eta(x_{i_1}) \int_{c_{i-1}}^{c_i} \varphi(x)\,dx = 0$$

Therefore,

$$\lim_{\epsilon \to 0} \frac{\Delta z}{\epsilon} = \frac{d}{d\epsilon} \{G[u(\xi) + \epsilon\varphi(\xi)]\}_{\epsilon=0} = \int_a^b G'[u(\xi);x]\varphi(x)\,dx \qquad (4)$$

Consequently, the differential dz which corresponds to the incremental function $du(\xi) = \epsilon\varphi(\xi)$ is defined to be

$$dz = \int_a^b G'[u(\xi);x]\epsilon\varphi(x)\,dx$$

From its derivation, it is seen that

$$\frac{dz}{\epsilon} = \lim_{\epsilon \to 0} \frac{\Delta z}{\epsilon}$$

so that dz differs from Δz by infinitesimals of higher order than ϵ, and hence satisfies the first of the two requirements placed on it. It also satisfies the linear requirement

$$dz[\epsilon_1\varphi_1(x) + \epsilon_2\varphi_2(x)] = \epsilon_1\, dz[\varphi_1(x)] + \epsilon_2\, dz[\varphi_2(x)]$$

since

$$\int_a^b G'[u(\xi);x]\left\{\epsilon_1\varphi_1(x) + \epsilon_2\varphi_2(x)\right\}dx = \epsilon_1 \int_a^b G'[u(\xi);x]\varphi_1(x)dx$$

$$+ \epsilon_2 \int_a^b G'[u(\xi);x]\varphi_2(x)dx$$

Consider now a functional $w = F\left[\underset{a}{\overset{b}{u(\xi)}},\underset{c}{\overset{d}{v(\xi)}}\right]$ with two arguments, and assume that both derivatives $F_u[u_o(\xi),v(\xi);x_1]$ and $F_v[u(\xi),v_o(\xi);x_2]$ exist for every x_1 and x_2 satisfying the inequalities $a < x_1 < b$ and $c < x_2 < d$. When $u(\xi)$ is given an increment $du(\xi) = \epsilon_1\varphi(\xi)$ while $v(\xi)$ is left unchanged, by extension of the above development the differential dw is

$$dw[\epsilon_1\varphi(\xi),0] = \int_a^b F_u[u(\xi),v(\xi);x]\epsilon_1\varphi(x)dx$$

Likewise, when $u(\xi)$ is unchanged and $v(\xi)$ is given an increment $dv(\xi) = \epsilon_2\psi(\xi)$, dw is

$$dw[0,\epsilon_2\psi(\xi)] = \int_c^d F_v[u(\xi),v(\xi);x]\epsilon_2\psi(x)dx$$

When the functions $u(\xi)$ and $v(\xi)$ are mutually independent, it follows that

$$dw[\epsilon_1\varphi(\xi),\epsilon_2\psi(\xi)] = dw[\epsilon_1\varphi(\xi),0] + dw[0,\epsilon_2\psi(\xi)]$$

Thus, the differential of a functional with two mutually independent arguments is given by

$$dw[\epsilon_1\varphi(\xi),\epsilon_2\psi(\xi)] = \epsilon_1 \int_a^b F_u[u(\xi),v(\xi);x]\varphi(x)dx$$

$$+ \epsilon_2 \int_c^d F_v[u(\xi),v(\xi);x]\psi(x)dx \qquad (5)$$

Integration of Step Responses

The preceding discussion provides the framework for the desired generalizations. Suppose that F is a variable whose value at a fixed point t is a functional of some physical quantity $u(\xi)$ defined in the interval $0 \leq \xi \leq t$; that is,

$$F(t) = G\left[\underset{o}{\overset{t}{u(\xi)}}\right]$$

In most physical problems, the argument function $u(\xi)$ is of such a nature that it can be built up as the limit of a sequence of step functions. For this reason it is desirable to develop an expression for $dF(t)$ when $du(\xi)$ is a step function. Consequently, the functions $u(\xi,\tau)$ and $\varphi(\xi,\tau)$ are defined as follows (cf. fig. 7):

$$\begin{aligned} u(\xi,\tau) &= u(\xi) \quad , \quad 0 \leq \xi < \tau \\ &= u(\tau) \quad , \quad \tau \leq \xi \leq t \\ \varphi(\xi,\tau) &= 0 \quad , \quad 0 \leq \xi < \tau \\ &= 1 \quad , \quad \tau \leq \xi \leq t \end{aligned}$$

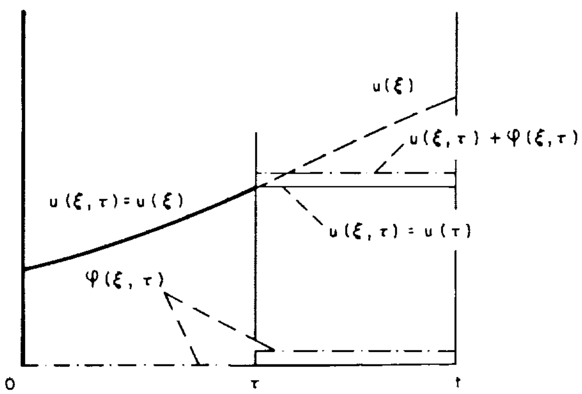


Figure 7.- Illustration of $u(\xi,\tau)$, $u(\xi)$, $\varphi(\xi,\tau)$.

When $u(\xi,\tau)$ is given an increment $du(\xi,\tau) = \epsilon\varphi(\xi,\tau)$, the resulting increment in F at the point t is

$$dF[\epsilon\varphi(\xi,\tau);t] = \epsilon \int_0^t G'[u(\xi,\tau);x]\varphi(x,\tau)\,dx$$

$$= \epsilon \int_\tau^t G'[u(\xi,\tau);x]dx$$

The nature of the integral

$$\int_\tau^t G'[u(\xi,\tau);x]dx$$

reveals the factors which influence the increment in F at the point t. Since the integrand G' is a functional of $u(\xi,\tau)$, it shows the increment in F to be a functional of $u(\xi,\tau)$. Thus the increment in F depends on all the values taken by $u(\xi,\tau)$ in the entire interval $0 \leq \xi \leq t$; but these values are all determined by the values taken by $u(\xi)$ in the subinterval $0 \leq \xi \leq \tau$. Consequently, the increment in F at the point t is a functional of $u(\xi)$ in the subinterval $[0,\tau]$; that is, it depends on the history of $u(\xi)$ in the period preceding the jump at $\xi = \tau$. Since an integral is a function of its limits, it is clear that the increment in F at t is a function of t and τ as well as a functional of $u(\xi)$. This function-functional dependence is expressed by the notation

$$F_u[u(\xi);t,\tau] = \int_\tau^t G'[u(\xi,\tau);x]dx \tag{6}$$

The subscript u is appended to emphasize that this integral is an extension of the notion of a derivative of a function. Indeed, equation (4) shows that

$$F_u[u(\xi);t,\tau] = \lim_{\epsilon\to 0} \frac{\Delta F}{\epsilon}$$

where ΔF is the increment in F corresponding to an increment $du(\xi)$. When this equation is compared with equation (1), that is,

$$C_{m_\alpha}(t - \tau) = \lim_{\Delta\alpha \to 0} \frac{\Delta C_m}{\Delta \alpha}(t - \tau)$$

the analogy is revealed between $F_u[u(\xi);t,\tau]$ and the indicial response appropriate to the linear case. Because $\epsilon F_u[u(\xi);t,\tau]$ is the differential increment in F at the point t which corresponds to a step increment in $u(\xi)$ at the point τ, and because $F_u[u(\xi);t,\tau]$ is analogous to the indicial response in the linear case, F_u will be called the generalized indicial response.

The generalized indicial response can be used to obtain an integral form for

$$F(t) = G[u(\xi)]$$

As in the linear case, the function $u(\xi)$, considered in the interval $[0,\tau+\Delta\tau]$, is approximated by $u(\xi)$ in the interval $[0,\tau]$ plus a step of height $\epsilon = (\Delta u/\Delta\tau)\Delta\tau$ somewhere in the interval $[\tau,\tau+\Delta\tau]$. As $\Delta\tau \to 0$, $\epsilon \to (du/d\tau)d\tau$ so that $dF(t) \to F_u[u(\xi);t,\tau](du/d\tau)d\tau$. Consequently,

$$\int_0^t dF = \int_0^t F_u[u(\xi);t,\tau]\, \frac{du}{d\tau}\, d\tau$$

or

$$F(t) = F(0) + \int_0^t F_u[u(\xi);t,\tau]\, \frac{du}{d\tau}\, d\tau \tag{7}$$

The integral on the right reduces to the familiar superposition integral when F and u are connected by a linear differential equation. Hence, the integral on the right of equation (7) is a generalization of the superposition integral.

An expression of similar character can be obtained when

$$F(t) = G\left[\overset{t}{\underset{0}{u(\xi)}},\overset{t}{\underset{0}{v(\xi)}}\right]$$

Reference to equation (5) shows

$$dF[\epsilon_1\varphi(\xi),\epsilon_2\psi(\xi);t] = \epsilon_1 \int_0^t G_u[u(\xi),v(\xi);x]\varphi(x)\,dx + \epsilon_2 \int_0^t G_v[u(\xi),v(\xi);x]\psi(x)\,dx \tag{8}$$

When $v(\xi)$ is a fixed function and $u(\xi)$ is given a step increment of height ϵ_1 at $\xi = \tau$, the response in F at point t is

$$dF[\epsilon_1\varphi(\xi,\tau),0;t] = \epsilon_1 \int_\tau^t G_u[u(\xi),v(\xi);x]dx$$

Likewise, when $u(\xi)$ is a fixed function and $v(\xi)$ is given a step increment of height ϵ_2 at $\xi = \tau$, the response in F at point t is

$$dF[0,\epsilon_2\psi(\xi,\tau);t] = \epsilon_2 \int_\tau^t G_v[u(\xi),v(\xi);x]dx$$

Accordingly, the expressions

$$F_u[u(\xi),v(\xi);t,\tau] = \int_\tau^t G_u[u(\xi),v(\xi);x]dx$$

and

$$F_v[u(\xi),v(\xi);t,\tau] = \int_\tau^t G_v[u(\xi),v(\xi);x]dx$$

are defined. Each is an indicial response. That is, $F_u[u(\xi),v(\xi);t,\tau]$ is the response in F at the point t to a unit step in $u(\xi)$ at $\xi = \tau$ when $v(\xi)$ is a fixed function. The response $F_v[u(\xi),v(\xi);t,\tau]$ is described similarly.

Consider now the response in F at $\xi = t$ to steps of height ϵ_1 and ϵ_2 made in $u(\xi)$ and $v(\xi)$, respectively, at $\xi = \tau$. Equation (8) and the notation just adopted give

$$dF = \epsilon_1 F_u[u(\xi),v(\xi);t,\tau] + \epsilon_2 F_v[u(\xi),v(\xi);t,\tau]$$

As before, $u(\xi)$ and $v(\xi)$ are approximated in the interval $[\tau,\tau+\Delta\tau]$ by steps of height $\epsilon_1 = (\Delta u/\Delta\tau)\Delta\tau$ and $\epsilon_2 = (\Delta v/\Delta\tau)\Delta\tau$, respectively. Again, as $\Delta\tau \to 0$, $\epsilon_1 \to (du/d\tau)d\tau$ and $\epsilon_2 \to (dv/d\tau)d\tau$ so that

$$dF \to F_u[u(\xi),v(\xi);t,\tau] \frac{du}{d\tau} d\tau + F_v[u(\xi),v(\xi);t,\tau] \frac{dv}{d\tau} d\tau$$

Therefore,

$$\int_0^t dF = \int_0^t F_u[u(\xi),v(\xi);t,\tau] \frac{du}{d\tau} d\tau + \int_0^t F_v[u(\xi),v(\xi);t,\tau] \frac{dv}{d\tau} d\tau$$

and, consequently,

$$F(t) = F(0) + \int_0^t F_u[u(\xi),v(\xi);t,\tau] \frac{du}{d\tau} d\tau + \int_0^t F_v[u(\xi),v(\xi);t,\tau] \frac{dv}{d\tau} d\tau \tag{9}$$

Equation (9) is the main result of this section. It reveals the structure of a functional with two argument functions in terms of generalized superposition integrals. In the next section, this result will be used in aerodynamic applications.

AERODYNAMIC APPLICATIONS

General Formulation for Aerodynamic Applications

It is clear that the preceding results provide the necessary mathematical framework for the proposed reformulation of the aerodynamic pitching-moment coefficient $C_m(t)$. Equation (6), which is free of dependence on a linearity assumption, generalizes the notion of indicial functions. Equation (7) is the corresponding generalization of the superposition integral. The relevance of these results to the aerodynamic problem under consideration is immediately evident when F, u, and v in equation (9) are replaced, respectively, by C_m, α, and q. The aerodynamic pitching-moment coefficient at time t is written in the general form

$$C_m(t) = C_m(0) + \int_0^t C_{m_\alpha}[\alpha(\xi),q(\xi);t,\tau]\,\frac{d}{d\tau}\,\alpha(\tau)d\tau$$

$$+ \int_0^t C_{m_q}[\alpha(\xi),q(\xi);t,\tau]\,\frac{d}{d\tau}\,q(\tau)d\tau \tag{10}$$

where $C_{m_\alpha}[\alpha(\xi),q(\xi);t,\tau]$ and $C_{m_q}[\alpha(\xi),q(\xi);t,\tau]$ are the generalized indicial responses, defined by direct analogy to the definitions of $F_u[u(\xi),v(\xi);t,\tau]$ and $F_v[u(\xi),v(\xi);t,\tau]$ given in the preceding section.

To attach a physical meaning to the definition of the indicial responses in terms appropriate to aerodynamic applications, the process of forming them will now be described. Two motions are considered: First, beginning at $\xi = 0$, the aircraft is caused to undergo the motion under study, $\alpha(\xi)$, $q(\xi)$. At a certain time τ, the motion is constrained in such a way that the values of the flow variables existent at time τ (i.e., $\alpha(\tau)$, $q(\tau)$) remain constant thereafter. The pitching moment corresponding to this maneuver is measured at a time t, subsequent to τ. Second, the aircraft is caused to execute precisely the same motion, beginning at $\xi = 0$ and constrained in the same way at $\xi = \tau$, except that at the latter time, one of the variables α or q is given an incremental step $\Delta\alpha$ or Δq over its value at $\xi = \tau$. Hence, if it is α which is given an increment $\Delta\alpha$, the values of the flow variables for all times subsequent to τ are $\alpha(\tau) + \Delta\alpha$, $q(\tau)$. The pitching moment is again measured at time t. The difference between the two measurements is divided by the incremental step $\Delta\alpha$ or Δq; the limit of this ratio as the magnitude of the step approaches zero is called the indicial pitching-moment response at time t per unit step at time τ of one of the two flow variables α or q. In the most general case, the indicial response depends not only on the levels $\alpha(\tau)$, $q(\tau)$ at which the step is made, but also on all the past values of α and q; in the most general case, then, the indicial response is itself a functional.

It will be noted that the form of equation (10) is remarkably similar to the exact linear form, equation (2). The resemblance is deceptive, however, for the redefined indicial responses are, in principle, far more complicated than their linear counterparts. Thus, for example, with the indicial responses dependent on the levels of α and q at which the steps are made, each successive response will be different from the last. In addition, the formulation allows for the possibility that the indicial responses may be functionals themselves, dependent on the whole past of α and q. Problems in which the latter condition must be retained may be so difficult as to lie outside the scope of the formulation's practicability. Fortunately, in a great many aerodynamic applications, the condition need not be retained, for it can be argued that under ordinary circumstances the indicial responses should have at most only a limited dependence on the past. The argument is presented in the next section.

Simplification of the General Formulation

In its full generality, equation (10) may provide at least a framework for the treatment of a wide variety of nonlinear aerodynamic problems, including, for example, those involving hysteresis phenomena. Here, however, the scope of equation (10) will be narrowed to a smaller class of problems, namely, that for which the indicial responses can be assumed to have only a limited dependence on the past. To demonstrate the nature of this assumption, it is convenient to adopt the two-dimensional wing as illustration. The conclusion, however, will apply to more general categories of wings and bodies.

Let the wing move away from a coordinate system whose origin is fixed in space at the center of gravity at a time $\xi = 0$, marking the beginning of a maneuver. The distance traveled by the center of gravity along the flight path is measured by a coordinate s. Since the center of gravity moves at constant velocity V_0, the trace of its path, plotted against time ξ, is a straight line. This is shown in figure 8, where the maximum projections of the leading and trailing edges are also shown, parallel to the trace of the center of gravity.

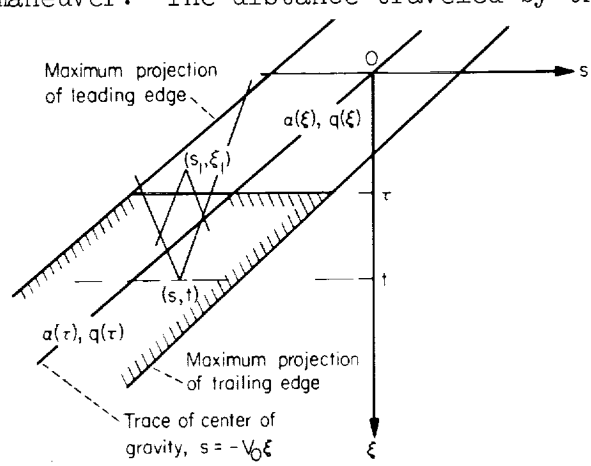


Figure 8.- Formation of indicial response.

Consider the two maneuvers prescribed for the formation of an indicial response. In both, the wing undergoes the motion under study $\alpha(\xi)$, $q(\xi)$ over the time interval zero to τ. For the first maneuver, in which the motion is constrained at τ so that the values of α and q existent at τ remain constant thereafter, consider a point s on the wing at a time t subsequent to τ. The loading at the point is influenced by all disturbances which originate in the past and are able to reach the point at the same time t. Each disturbance is propagated at the local speed of sound, and hence, in a plot such as figure 8, the zone of its influence is bounded by projections of the rays of an approximately conic surface whose origin is the point of the disturbance. Only disturbances whose cones include the point (s,t) in question can influence the loading at the point. Hence, a certain conic surface, directed backward in time from the point (s,t), will include within it all points in past time whose disturbances are able to influence the loading at s at time t. Such a cone is shown in figure 8 for a case where the flight speed V_0 is supersonic. It will be noted that both rays of the cone intersect the leading edge; since at supersonic speed, there are no disturbances in the region ahead of the shock wave which emanates from the leading edge, it is clear that the zone of disturbances enclosed by the cone is bounded in time and space. Hence, for supersonic speed at least, it is already evident that the past events which can influence the indicial loading at time t are limited to those occurring within a definite time interval of the most recent past.

Now consider the same point s at time t during the second of the two motions. Since the motions prior to τ are identical, the propagation of a disturbance originating prior to τ at a point (s_1,ξ_1) will be precisely the same over the time interval $\tau - \xi_1$ as that of its counterpart during the first motion. Hence, the disturbance will arrive at the same point at time τ and in the same condition. Then, if it arrives at the point s at a different time or in a different condition than its counterpart, the modification must be due to its interaction with disturbances which originate <u>after</u> time τ. Now the latter disturbances themselves differ between the two motions by terms of $O(\Delta\alpha)$ or $O(\Delta q)$. Then it is reasonable to assume that <u>interactions</u> between disturbances normally will differ in the two cases by terms in $\Delta\alpha$ or Δq of order higher than the first. The assumption is, then, that the influence on the loading at point s of a disturbance which originates at a time prior to τ during the second maneuver will be, within terms of $O((\Delta\alpha)^2)$ or $O((\Delta q)^2)$, identical to that of its counterpart during the first maneuver. Therefore, with terms neglected of order higher than the first in $\Delta\alpha$ or Δq, the influence of events prior to τ will cancel in the formation of the indicial response. It is believed that the assumption should hold so long as the incremental change $\Delta\alpha$ or Δq does not introduce a radical change in the nature of the flow field, such as a new or greatly altered shock wave or an abrupt shift in the pattern of flow separation.

The above assumption has several important consequences. First, since within the assumption the indicial response is insensitive to the motion prior to τ, that motion may be assigned as desired. It is convenient to specify that it be invariant with time; that is, for all $\xi < \tau$, let it be specified that $\alpha = \alpha(\tau)$, $q = q(\tau)$. The indicial response, now dependent on the <u>parameters</u> $\alpha(\tau)$, $q(\tau)$, rather than on the functions $\alpha(\xi)$, $q(\xi)$, may be considered a function in the ordinary sense rather than a functional. To indicate this specifically, the following notation is adopted for the indicial functions

$$\left.\begin{aligned} C_{m_\alpha}[\alpha(\xi),q(\xi);t,\tau] &= C_{m_\alpha}(t,\tau;\alpha(\tau),q(\tau)) \\ C_{m_q}[\alpha(\xi),q(\xi);t,\tau] &= C_{m_q}(t,\tau;\alpha(\tau),q(\tau)) \end{aligned}\right\} \tag{11}$$

Although equations (11) have been obtained under the assumption that events prior to τ do not influence the indicial response, the assignation of a time-invariant past to the response renders the assumption substantially less restrictive. Thus, in certain cases, even if events prior to τ do in fact influence the indicial response, equations (11) may still apply approximately. This will be true, for example, when the flight speed is supersonic and the motion under consideration is slowly varying. The first of these provisions ensures that only a limited interval of the most recent past can affect the indicial response, while the second ensures that, over this limited interval, the values of α and q change only slightly. Hence, so far as the indicial response is cognizant of the past motion, the past motion is essentially time-invariant.

A second consequence of the assumption (and not an additional assumption) can be deduced when it is recalled that the flight speed and atmospheric density also remain invariant throughout the motion. It is easy to show that the indicial response for given values of the parameters $\alpha(\tau)$ and $q(\tau)$ will be the same at a given time subsequent to the step no matter when the step occurs. This means that the response cannot be a function of t and τ separately, but must be a function only of elapsed time subsequent to the step, that is, a function of the time difference $t - \tau$. Let it be noted, however, that this assertion holds only by virtue of the prescribed constancy of flight conditions along the flight path. It would not hold, for example, for accelerated motions, or for atmospheric entry motions where the atmospheric density must be considered a variable.

Finally, then, a much more specific form of equation (10) may be written, which seems capable of embracing a fairly broad range of aerodynamic problems. It is

$$C_m(t) = C_m(0) + \int_0^t C_{m_\alpha}(t-\tau;\alpha(\tau),q(\tau))\,\frac{d}{d\tau}\,\alpha(\tau)\,d\tau + \int_0^t C_{m_q}(t-\tau;\alpha(\tau),q(\tau))\,\frac{d}{d\tau}\,q(\tau)\,d\tau \tag{12}$$

Although the form of equation (12) represents a great simplification over that of equation (10), it is worth noting that the equation still includes the full linear form (eq. (2)) as a special case.

Application to Dynamic Stability Studies

Equation (12) is now applied to the study of aircraft dynamic stability. The rigid-body motions of aircraft are normally oscillatory, and moreover, the oscillations are generally of very low frequency. Several analytical benefits accrue from the latter fact: First, since the motions are slowly varying, the assumptions underlying equation (12) are particularly well-grounded in this application. Second, equation (12) can be further simplified. The simplification, which in effect reduces equation (12) to an equation correct to the first order in frequency, parallels that realized in the linear case in the application of equation (2) to stability studies (ref. 4).

Equation (12) is first rearranged to give a more convenient form. It is evident from physical considerations that the indicial functions will approach or reach steady-state values with increasing values of the argument $t - \tau$. To indicate this, the following identities are introduced (the notation parallels that of ref. 4):

$$\left.\begin{aligned} C_{m_\alpha}(t-\tau;\alpha(\tau),q(\tau)) &= C_{m_\alpha}(\infty;\alpha(\tau),q(\tau)) - F_3(t-\tau;\alpha(\tau),q(\tau)) \\ C_{m_q}(t-\tau;\alpha(\tau),q(\tau)) &= C_{m_q}(\infty;\alpha(\tau),q(\tau)) - F_4(t-\tau;\alpha(\tau),q(\tau)) \end{aligned}\right\} \tag{13}$$

where

$C_{m_\alpha}(\infty;\alpha(\tau),q(\tau))$ rate of change with α of the pitching-moment coefficient that would be measured in a steady flow, evaluated at the instantaneous value $\alpha(\tau)$ with q fixed at the instantaneous value $q(\tau)$

$C_{m_q}(\infty;\alpha(\tau),q(\tau))$ rate of change with q of the pitching-moment coefficient that would be measured in a steady flow, evaluated at the instantaneous value $q(\tau)$ with α fixed at the instantaneous value $\alpha(\tau)$

The functions F_3 and F_4 are termed deficiency functions; they of course tend to vanish with increasing values of the argument $t - \tau$. When equations (13) are inserted in equation (12), the terms involving the steady-state parameters form a perfect differential which can be immediately integrated. Equation (12) becomes

$$\begin{aligned} C_m(t) = C_m(\infty;\alpha(t),q(t)) &- \int_0^t F_3(t-\tau;\alpha(\tau),q(\tau)) \frac{d}{d\tau}\alpha(\tau)d\tau \\ &- \int_0^t F_4(t-\tau;\alpha(\tau),q(\tau)) \frac{d}{d\tau} q(\tau)d\tau \end{aligned} \tag{14}$$

where

$C_m(\infty;\alpha(t),q(t))$ total pitching-moment coefficient that would be measured in a steady flow with α fixed at the instantaneous value $\alpha(t)$ and q fixed at the instantaneous value $q(t)$

Equation (14) is a form of equation (12) particularly amenable to approximation. It is desired to reduce the equation to one that is correct to the first order in frequency. Let it be assumed for illustration that the angles α and θ are essentially harmonic functions, that is,

$$\alpha \approx \alpha_0 e^{i\omega t}$$
$$\theta \approx \theta_0 e^{i\omega t}$$

Then it is clear that, since q is proportional to $\dot{\theta}$, q itself will be of first order in frequency ω. Hence, q will be small for all values of time, and powers of q higher than the first will be of second and higher orders in frequency. Therefore, for any given values of t or τ, it is permissible to expand the terms in equation (14) in a Taylor series about $q = 0$ and to discard terms containing powers of q higher than the first. It is clear also that terms in $\dot{q}$ and $\dot{\alpha}q$ likewise may be discarded as they will be of second order in frequency. After the expansion is carried out and terms are discarded as described, the result is

$$C_m(t) = C_m(\infty;\alpha(t),0) + q(t)C_{m_q}(\infty;\alpha(t),0) - \int_0^t F_3(t-\tau;\alpha(\tau),0)\,\frac{d}{d\tau}\,\alpha(\tau)\,d\tau \tag{15}$$

Definitions of $C_m(\infty;\alpha(t),0)$ and $C_{m_q}(\infty;\alpha(t),0)$ follow from those given previously with the substitution $q(t) = 0$. The first two terms are clearly the nonlinear counterparts of the terms $\alpha C_{m_\alpha}(\infty)$ and $qC_{m_q}(\infty)$ that appear in linear analyses based on the stability derivative concept. It is to be expected, therefore, that the integral, when also reduced to the first order in frequency, should be the nonlinear counterpart of the term $(\dot{\alpha}l/V_o)C_{m_{\dot{\alpha}}}$. This reduction is considered next.

The reduction of the integral is most conveniently demonstrated when the assumption of a harmonic variation in α is retained. It should be understood, however, that the argument does not hinge on this assumption. With the change in variable $t - \tau = \tau_1$, the integral becomes

$$I = \int_0^t F_3(\tau_1;\alpha(t-\tau_1),0)\,i\omega\alpha_o e^{i\omega(t-\tau_1)}\,d\tau_1 \tag{16}$$

which may be rewritten

$$I = \dot{\alpha}(t)\int_0^t F_3(\tau_1;\alpha(t-\tau_1),0)\,e^{-i\omega\tau_1}\,d\tau_1 \tag{17}$$

Now for supersonic speed, the deficiency function vanishes after a finite and relatively short interval of time τ_1 has elapsed (the interval diminishes with increasing speed). For subsonic speed, in theory the deficiency function approaches zero only asymptotically as $\tau_1 \to \infty$, but it is reasonable to assume that in practice the difference from zero soon would be unmeasurable; hence, here also, the deficiency function may be said to vanish after a relatively short period of time. Let the value of τ_1 at which the deficiency function essentially vanishes be t_a, and consider events at a time t sufficiently removed from the start of the motion that $t \gg t_a$. Then the upper limit in equation (17) may be replaced by t_a, whereupon it is clear that with τ_1 bounded and ω small, the harmonic function may be expanded in powers of ω. Since $\dot{\alpha}$ is itself of first order in ω, however, only the first term in the expansion, unity, contributes within the order of the approximation. Moreover, with respect to the parameter $\alpha(t - \tau_1)$, a further simplification can be realized when the condition $t \gg t_a$ is invoked, for then $\alpha(t - \tau_1) \approx \alpha(t)$. The integral reduces to

$$I = \frac{\dot{\alpha}(t)\,l}{V_o}\left(\frac{V_o}{l}\int_0^{t_a} F_3(\tau_1;\alpha(t),0)\,d\tau_1\right) \tag{18}$$

Hence, just as in the linear case (cf. ref. 4), to the first order in frequency the integral is essentially the area of the deficiency function, now, however, evaluated at and dependent on the particular value of angle of attack $\alpha(t)$ under consideration. Further, it can be demonstrated that the analogy shown to exist in the linear case between the area of the deficiency function and the stability derivative $C_{m_{\dot{\alpha}}}$ (ref. 4) carries over as well (on condition that $t \gg t_a$). The form of equation (12) appropriate to dynamic stability analyses is then

$$C_m(t) = C_m(\infty;\alpha(t),0) + q(t)C_{m_q}(\infty;\alpha(t),0) + \dot{\alpha}(t)\frac{l}{V_o}C_{m_{\dot{\alpha}}}(\alpha(t)) \qquad (19)$$

where

$$C_{m_{\dot{\alpha}}}(\alpha(t)) = -\frac{V_o}{l}\int_0^{t_a} F_3(\tau_1;\alpha(t),0)d\tau_1 \qquad (20)$$

Equation (19) may be given a more uniform appearance by the adoption of a dimensionless measure φ as the independent variable in place of time t. Let

$$\left.\begin{aligned} \varphi &= \frac{V_o t}{l}, \quad \text{number of lengths traveled in time } t \\ \varphi_1 &= \frac{V_o \tau_1}{l}, \quad \text{number of lengths traveled in time } \tau_1 \\ \sigma_a &= \frac{V_o t_a}{l}, \quad \text{number of lengths traveled in time required} \\ &\qquad \text{for deficiency function to vanish} \end{aligned}\right\} \qquad (21)$$

Equation (19) becomes

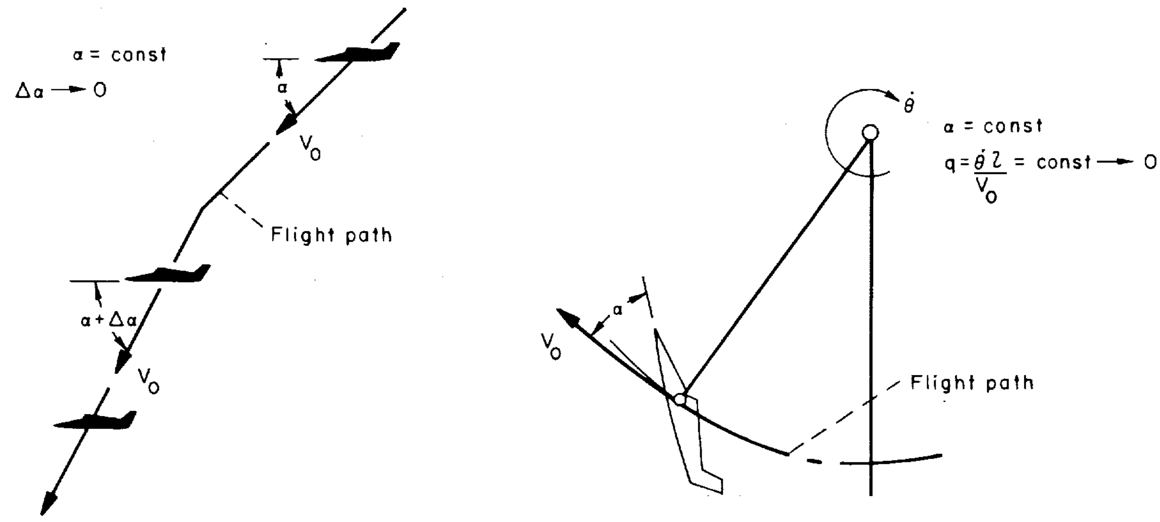
$$C_m(\varphi) = C_m(\infty;\alpha(\varphi),0) + q(\varphi)C_{m_q}(\infty;\alpha(\varphi),0) + \alpha'(\varphi)C_{m_{\dot{\alpha}}}(\alpha(\varphi)) \qquad (22)$$

with

$$C_{m_{\dot{\alpha}}}(\alpha(\varphi)) = -\int_0^{\sigma_a} F_3(\varphi_1;\alpha(\varphi),0)d\varphi_1 \qquad (23)$$

Of the three coefficients in equation (22), the first, of course, is the familiar static pitching-moment coefficient due to angle of attack. Since it is of fundamental importance in aircraft design, it has been studied extensively. Consequently, a large body of relevant experimental results exists which may be called on to define the coefficient in particular cases. On the contrary, no such body of results exists for the remaining two coefficients. It is recalled, however, that the two terms arise as the result of small perturbations in the flow variables from a possibly large but fixed angle of attack. This fact suggests the possibility that the terms may be obtainable, at least in certain cases, from theoretical calculations. That is, it may be possible to adapt the methods of small-disturbance theory to this problem, where, it is anticipated, the perturbation equations will have coefficients dependent on the basic flow field corresponding to the fixed angle of attack.

Just as in the linear case, solutions for C_{m_q} may be derived from a steady-state equation, whereas solutions for $C_{m_{\dot{\alpha}}}$ will have to be derived from a time-dependent equation. The specific problems which must be solved are characterized by the motions illustrated in figure 9.



(a) Motion for $F_3(\varphi_1;\alpha,0)$.

(b) Motion for $C_{m_q}(\infty;\alpha,0)$.

Figure 9.- Definitive motions for calculation of stability coefficients.

It remains to discuss the significance of the integral form for $C_{m_{\dot{\alpha}}}$, equation (23). This requires a short preliminary discussion of the indicial function, and to this end, the two-dimensional wing is again used as illustration; again, however, the results will have more general bearing. Consider the boundary conditions corresponding to the motion illustrated in figure 9(a) for a supersonic flight speed. As shown in figure 10, for $\varphi_1 < 0$ the angle of attack is constant at $\alpha(\varphi)$. A step in α occurs at $\varphi_1 = 0$ so that for $\varphi_1 \geq 0$, the angle of attack is $\alpha(\varphi) + \Delta\alpha$. Due to the impulsive change in α, the loading on the wing at $\varphi_1 = 0$ likewise undergoes a sudden change. The physical situation at this instant corresponds essentially to that described by piston theory (cf., e.g., ref. 6), and indeed, this theory should yield a reasonably accurate estimate of the initial change in loading. Also at $\varphi_1 = 0$, the sudden change in flow conditions causes a sound wave to propagate from the leading edge and, as shown in figure 10, the wave divides the wing into three distinct regions. Points in region (1)

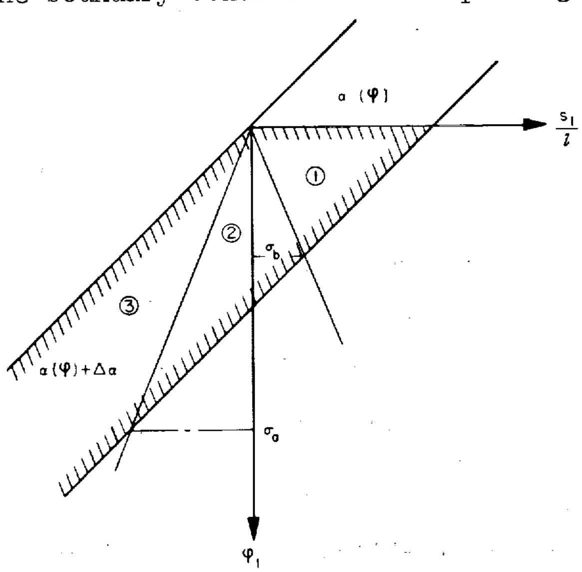


Figure 10.- Boundary conditions for the indicial loading.

have not yet been made aware of the changed conditions at the leading edge by the arrival of the sound wave, and hence the loading in this region persists essentially unchanged from that existent at $\varphi_1 = 0$. This loading gradually disappears as the propagation of the sound wave announces the new conditions at the leading edge to increasing portions of the wing, and it disappears completely at $\varphi_1 = \sigma_b$. Region (3) sets off that portion of the wing which has outrun the wave and which has therefore already acquired the steady-state loading corresponding to the new angle of attack. The entire wing attains this state at $\varphi_1 = \sigma_a$. Region (2) is an intermediate regime, under the direct influence of the sound wave. In accordance with the above behavior, there is reason to break the indicial pitching-moment function into two separate contributions, as shown in figure 11. The first variation represents the pitching-moment contribution of the integrated loading in region (1); accordingly, it begins with the value $C_{m_\alpha}(0;\alpha(\varphi),0)$ and vanishes at $\varphi_1 = \sigma_b$. The second variation reflects the lumped contributions of regions (2) and (3); in conformity with the loading, its initial value is zero, while its end value, attained at $\varphi_1 = \sigma_a$, is $C_{m_\alpha}(\infty;\alpha(\varphi),0)$. The sum of the two variations is the indicial pitching-moment function $C_{m_\alpha}(\varphi_1;\alpha(\varphi),0)$. In order to exhibit the end values of the two contributions explicitly, the normalized functions f_1 and f_2 are introduced as shown in the figure, whereupon the indicial response is written in the form

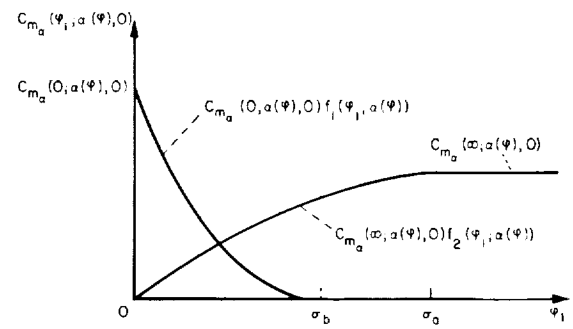


Figure 11.- Breakdown of indicial response.

$$C_{m_\alpha}(\varphi_1;\alpha(\varphi),0) = C_{m_\alpha}(0;\alpha(\varphi),0)f_1(\varphi_1;\alpha(\varphi)) + C_{m_\alpha}(\infty;\alpha(\varphi),0)f_2(\varphi_1;\alpha(\varphi)) \tag{24}$$

where f_1 and f_2 vary within the limits zero and unity. Hence, the deficiency function F_3 is

$$F_3(\varphi_1;\alpha(\varphi),0) = C_{m_\alpha}(\infty;\alpha(\varphi),0)[1 - f_2(\varphi_1;\alpha(\varphi))] - C_{m_\alpha}(0;\alpha(\varphi),0)f_1(\varphi_1;\alpha(\varphi)) \tag{25}$$

so that the integral for $C_{m_{\dot\alpha}}$ takes the form

$$\begin{aligned} C_{m_{\dot\alpha}}(\alpha(\varphi)) = {} & C_{m_\alpha}(0;\alpha(\varphi),0) \int_0^{\sigma_b} f_1(\varphi_1;\alpha(\varphi))d\varphi_1 \\ & - C_{m_\alpha}(\infty;\alpha(\varphi),0) \int_0^{\sigma_a} [1 - f_2(\varphi_1;\alpha(\varphi))]d\varphi_1 \end{aligned} \tag{26}$$

Equation (26) clearly shows the strong dependence of $C_{m_{\dot\alpha}}$ on the steady-state parameter $C_{m_\alpha}(\infty;\alpha(\varphi),0)$. Further, the sign relation evidenced in equation (26) should be noted. Since the integral involving f_2 normally will be positive, the equation indicates that as $C_{m_\alpha}(\infty;\alpha(\varphi),0)$ becomes more negative (i.e., statically more stable), $C_{m_{\dot{\alpha}}}$ becomes more positive (i.e., dynamically more unstable). This has been observed to be the case in a number of experimental investigations (cf. ref. 7 and references cited therein).

Since f_1 and f_2 are normalized functions, cases are anticipated in which their dependence on α will not be significant. In such cases, the integrals in equation (26) become constants. When it can be further assumed that $C_{m_\alpha}(0;\alpha(\varphi),0)$ is also independent of α, the expression for $C_{m_{\dot{\alpha}}}$ takes the simple form

$$C_{m_{\dot{\alpha}}}(\alpha(\varphi)) = A + BC_{m_\alpha}(\infty;\alpha(\varphi),0) \tag{27}$$

A version of this result, applicable to the case of small harmonic oscillations about a mean value of α, was first presented in reference 7. Equation (27) is the more general form, however, within the same assumptions. The most general form of the relation is, of course, equation (26).

Implications for Experiments

The increasing appearance of nonlinear aerodynamic phenomena has been of particular concern to experimenters. A great many facilities and techniques have been developed for extracting stability coefficients from dynamical data expressly on the assumption that the motions were governed by linear differential equations having constant coefficients. The widely different methods which have been developed on this assumption in principle yield identical results when the assumption is valid, but there is no such assurance when it is not. In the latter case, a different result may be obtained from each facility and from each method of data reduction, each result reflecting a particular facet and usually some linear "equivalent" of the underlying nonlinear phenomenon. The analyst who must use these results in a particular application is often faced with many choices; he can be assured of the validity of his choice only if the motion he anticipates in his application bears some resemblance to the motion from which the results chosen were extracted. Obviously, this situation imposes a severe limitation on the usefulness of wind-tunnel measurements of stability coefficients.

The formulation presented in the preceding section may be addressed to this problem in two ways: first, with regard to a particular facility and model, to establish the <u>form</u> of the equation governing the motion, and second, with regard to results from various facilities and an identical model, to seek <u>correlations</u> between the results which might widen the range of motions for which the results could be made to apply interchangeably. Consider the first of these uses for the case of wind-tunnel dynamical experiments. As is usual in such experiments, let the model be pinned at its center of gravity to a fixed point, so that it executes purely rotary motions about the center of gravity. In this case the angle of pitch θ and the angle of attack α are the same. Consequently, with the dimensionless measure φ as independent variable, $q(\varphi) = \alpha'(\varphi)$. Then the most general form of equation (22) is

$$C_m(\varphi) = \alpha'(\varphi)h(\alpha(\varphi)) + g(\alpha(\varphi)) \tag{28}$$

where

$$h(\alpha(\varphi)) = C_{m_q}(\infty;\alpha(\varphi),0) + C_{m_{\dot{\alpha}}}(\alpha(\varphi))$$

$$g(\alpha(\varphi)) = C_m(\infty;\alpha(\varphi),0)$$

The dependence on α alone of the damping coefficient h should be noted. That it is independent of $\alpha'(\varphi)$ or higher derivatives of α is a necessary and consistent consequence of the restriction that the motion be slowly varying. Further specifications of the form of h and g will depend on the particular model under study. One such specification of $C_{m_{\dot{\alpha}}}$ already has been presented as equation (27). A similar form for h may be specified under a number of different circumstances. Perhaps the most general of these occurs when it is known that g is adequately represented by the first two terms of an expansion in odd powers of α. Since the boundary conditions for the motion from which $C_{m_q}(\infty;\alpha(\varphi),0)$ is derived are similar to those for $C_{m_\alpha}(\infty;\alpha(\varphi),0)$, it is very reasonable to assume that C_{m_q}, like C_{m_α}, will show no more than an even quadratic dependence on α. Further, reference 6 shows that for many cases $C_{m_\alpha}(0;\alpha(\varphi),0)$ will be expressible as a quadratic in α. Under similar conditions, the functions f_1 and f_2 in equation (26), being normalized, may be assumed to be essentially independent of α. Then h and g take the forms

$$\left.\begin{aligned} h &= h_o + h_2\alpha^2 \\ g &= \alpha(g_o + g_2\alpha^2) \end{aligned}\right\} \tag{29}$$

Alternatively, therefore, h is expressible in the form previously exhibited as a particularly simple one for $C_{m_{\dot{\alpha}}}$ (eq. (27)), namely

$$h = C_{m_q} + C_{m_{\dot{\alpha}}} = C + DC_{m_\alpha}(\infty;\alpha(\varphi),0) \tag{30}$$

Since a limited range of α normally will exist over which g can be expressed as a cubic, equation (30) would appear to have general applicability at least in that range. For much wider ranges of α, this simple relation probably will not hold. The generalization of the above result leads rather to a prediction that if g is expressible as an odd polynomial in α of degree n, then h will be an even polynomial in α of degree $n - 1$.

The second of the two uses, correlation, is now illustrated for a model whose coefficients are assumed to be of the form given by equations (29) and (30). Two widely used experimental methods are considered. The first is illustrated in figure 12. The sting is brought to a fixed mean angle of

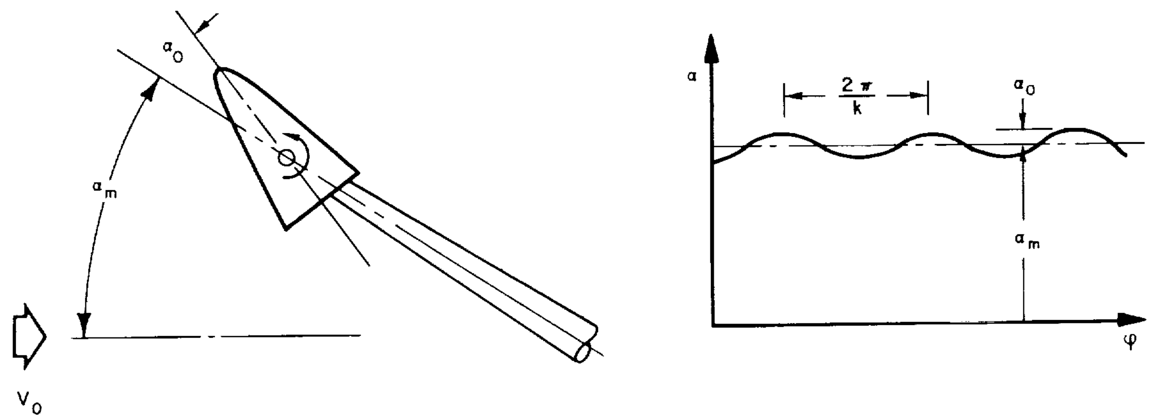


Figure 12.- Forced oscillations about a mean angle of attack.

attack α_m and the model is forced to oscillate periodically about α_m with a small amplitude α_o. The work required to drive the model (in excess of a known wind-off tare) is measured and equated to the work done by the aerodynamic damping moment. Thus

$$\frac{\text{work}}{\text{cycle}} = W = q_o S l \int_0^{2\pi/k} \alpha'^2(\varphi)[C_{m_q} + C_{m_{\dot{\alpha}}}]d\varphi \tag{31}$$

where

$$k = \frac{\omega l}{V_o} \quad , \qquad \text{reduced frequency}$$

Let α be a harmonic function about the mean angle α_m; that is,

$$\alpha(\varphi) = \alpha_m + \alpha_o \sin k\varphi \tag{32}$$

In the linear case, where the damping coefficient is a constant, integration of equation (31) yields

$$\frac{W}{q_o S l \alpha_o{}^2 \pi k} = C_{m_q} + C_{m_{\dot{\alpha}}} \tag{33}$$

Thus, the damping coefficient is obtained in terms of measurable quantities. Now let the damping coefficient be nonlinear and of the form given by equation (30). Equation (30) is substituted in (31) and $C_{m_\alpha}(\infty;\alpha,0)$ is expanded in a power series about α_m. The following shortened notation is adopted in the expansion.

$$C_{m_\alpha}(\infty;\alpha(\varphi),0) = C_m'(\infty;\alpha_m) + (\alpha - \alpha_m)C_m''(\infty;\alpha_m) + \frac{(\alpha - \alpha_m)^2}{2!} C_m'''(\infty;\alpha_m) + \ldots \tag{34}$$

When equation (32) is substituted in (34), the integrations can be carried out. Forming the ratio indicated in equation (33) then yields an "effective" damping coefficient

$$\frac{W}{q_o S l \alpha_o^2 \pi k} = (C_{m_q} + C_{m_{\dot{\alpha}}})_e$$
$$= C + D\left[C_m'(\infty;\alpha_m) + \frac{\alpha_o^2}{8} C_m'''(\infty;\alpha_m) + \ . \ . \ .\right] \tag{35}$$

Comparison of equation (35) with equation (30) reveals the significance of the bracketed term. Just as the measured damping coefficient is an "effective" value of $C_{m_q} + C_{m_{\dot{\alpha}}}$, the bracketed quantity is an effective value of the static pitching-moment curve slope $C_{m_\alpha}(\infty;\alpha,0)$. It is, in fact, the value that would be obtained from frequency measurements were the apparatus to be tuned to oscillate at its resonant frequency (cf. ref. 8). Hence, equation (35) can be written as

$$(C_{m_q} + C_{m_{\dot{\alpha}}})_e = C + DC_{m_{\alpha_e}} \tag{36}$$

with

$$C_{m_{\alpha_e}} = C_m'(\infty;\alpha_m) + \frac{\alpha_o^2}{8} C_m'''(\infty;\alpha_m) + \ . \ . \ . \tag{37}$$

Let the experiment be repeated for a range of values of α_m and a given value of α_o. Equation (36) indicates that if the results for $(C_{m_q} + C_{m_{\dot{\alpha}}})_e$ are plotted against $C_{m_{\alpha_e}}$, a straight line is obtained. Further the same straight line is obtained for any other value of α_o. Hence, in effect, equation (36) correlates the data.[2] The zero-intercept and slope of the line give the values of C and D, respectively. When these values are inserted in equation (30), the resulting expression for $C_{m_q} + C_{m_{\dot{\alpha}}}$ is free of dependence on the particular method used to measure the damping; it is, in other words, applicable to <u>any</u> motion satisfying the original restrictions (i.e., purely longitudinal variations at constant flight speed).

The second method is the well-known free-oscillation technique. As shown in figure 13, the model is displaced from zero angle of attack against

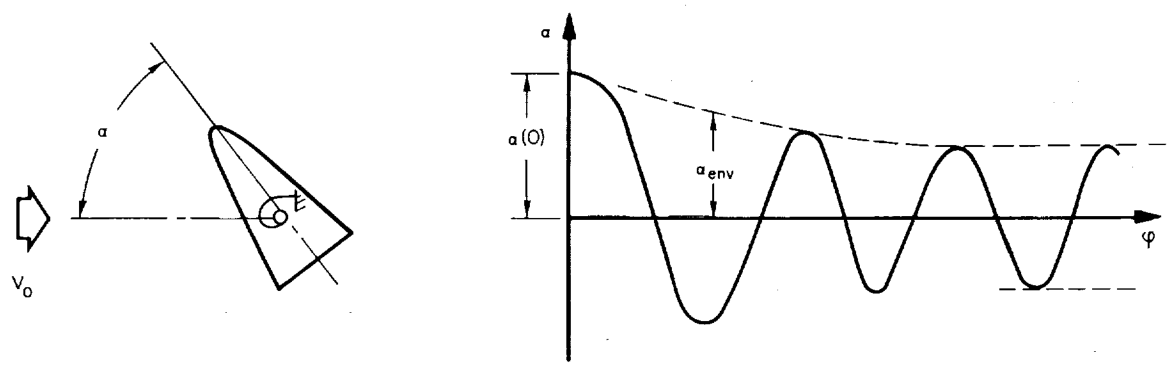


Figure 13.- Free-oscillation technique.

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[2]A version of this result, applicable under somewhat different circumstances, was first presented in reference 7 together with a qualitative indication of an experimental confirmation.

the resistance of a spring, and then released. The resulting free oscillation is recorded. With the model the same as that considered above, the aerodynamic pitching-moment coefficients are again given by equations (29) and (30). The equation of motion, with φ as independent variable, is therefore

$$\left(\frac{2I}{\rho_o S l^3}\right) \alpha''(\varphi) = (h_o + \bar{h}_o + h_2\alpha^2)\alpha'(\varphi) + (g_o + \bar{g}_o + g_2\alpha^2)\alpha(\varphi) \tag{38}$$

where the damping and restoring-moment coefficients of the spring, assumed constant and known, are indicated as $\bar{h}_o$ and $\bar{g}_o$. Hence, the characteristic equation is a combination of Duffing's equation and the Van der Pol equation. Note that within the latter form the possibility exists of a limit motion. As this is the interesting case, let it occur. Let

$$\left.\begin{aligned} c^2 &= \left(\frac{\rho_o S l^3}{2I}\right)(h_o + \bar{h}_o) \\ d^2 &= -\left(\frac{\rho_o S l^3}{2I}\right) h_2 \\ e^2 &= -\left(\frac{\rho_o S l^3}{2I}\right)(g_o + \bar{g}_o) \\ f^2 &= \left(\frac{\rho_o S l^3}{2I}\right) g_2 \end{aligned}\right\} \tag{39}$$

and change the dependent variable to

$$\lambda = \frac{d}{c}\,\alpha \tag{40}$$

Equation (38) becomes

$$\lambda''(\varphi) - c^2\lambda'(\varphi)(1 - \lambda^2) + e^2\lambda\left(1 - \frac{f^2c^2}{d^2e^2}\,\lambda^2\right) = 0 \tag{41}$$

The change in independent variable

$$y = e\varphi \tag{42}$$

and the substitutions

$$\left.\begin{aligned} \frac{c^2}{e} &= \epsilon \\ \frac{f^2c^2}{d^2e^2} &= \epsilon\mu \end{aligned}\right\} \tag{43}$$

bring equation (41) to

$$\lambda''(y) - \epsilon\lambda'(y)(1 - \lambda^2) + \lambda(1 - \epsilon\mu\lambda^2) = 0 \tag{44}$$

With $\epsilon > 0$, equation (44) indicates that a limit motion will occur. The amplitude of this motion is approximately $\lambda_{env} \approx 2$. For an initial value λ_o greater than 2, the amplitude will converge to 2 and the oscillation will sustain itself at that amplitude thereafter. For $\lambda_o < 2$, the amplitude will diverge to the same limit amplitude. An approximate solution which shows this behavior, adequate for small ϵ, can be obtained by means of the Kryloff-Bogoliuboff method (ref. 8). The solution is

$$\lambda = \frac{2}{\sqrt{1 - e^{-\epsilon y}\left[1 - (4/\lambda_o^2)\right]}} \sin\left\{y - \frac{3}{2}\mu \log\left[1 + \frac{\lambda_o^2}{4}\left(e^{\epsilon y} - 1\right)\right] + \psi_o\right\} \tag{45}$$

The envelope of the oscillation is

$$\lambda_{env} = \frac{2}{\sqrt{1 - e^{-\epsilon y}\left[1 - (4/\lambda_o^2)\right]}} \tag{46}$$

which, as described, approaches 2 for $y \to \infty$. Since, for large φ, α_{env} will be constant and can be measured from the data, the relation

$$2 = \frac{d}{c}\left(\alpha_{env}\right)_{\varphi\to\infty} \tag{47}$$

determines the scale factor between λ and α in equation (40). Now return to the original variable φ and rewrite equation (46) as

$$\log\left(\frac{\lambda_{env}^2}{\lambda_{env}^2 - 4}\right) = \log\left(\frac{\lambda_o^2}{\lambda_o^2 - 4}\right) + c^2\varphi \tag{48}$$

If the left side is plotted against φ on semilogarithmic paper, a straight line is obtained whose slope is c^2. This determines c, whereupon d is determinable from equation (47).[3] The constants e and f may be obtained from measurements of the change in half-period between successive zeros of the oscillation. The four constants h_o, h_2, g_o, g_2 are then determinable from equations (39). When the damping coefficient is written in the form of equation (30)

$$C_{m_q} + C_{m_{\dot{\alpha}}} = h_o + h_2\alpha^2 = C + DC_{m_\alpha}(\infty;\alpha(\varphi),0) \tag{49}$$

and $C_{m_\alpha}(\infty;\alpha(\varphi),0)$ is inserted from equation (29), the following identities are obtained:

$$\left.\begin{aligned} h_o &= C + Dg_o \\ h_2 &= 3g_2D \end{aligned}\right\} \tag{50}$$

[3]The authors are indebted to Mr. Peter J. Mantle for pointing out this interpretation of the data.

Hence, C and D may be calculated. The latter values should be the same as those obtained in the first experiment.

The above discussion suggests an obvious experiment: Select a model whose steady-state pitching-moment coefficient $C_m(\infty;\alpha,0)$ is known to be representable as a cubic in α over a substantial range of angle of attack. Carry out the two experiments in the manner described above. If the results for $C_{m_q} + C_{m_{\dot{\alpha}}}$ are expressible in the form of equation (30) and if the values of C and D agree between the two experiments, this will count both as a confirmation of the theory and a useful means of widening the applicability of the experimental results.

CONCLUDING REMARKS

A theoretical study has been undertaken of nonlinear longitudinal dynamic stability. The mathematical theory of functionals was adapted to serve as the framework for a reformulation of the notions of aerodynamic indicial functions and superposition. This led to the derivation of an integral form for the aerodynamic pitching-moment coefficient which is free of dependence on a linearity assumption. Applications of the results to theoretical and experimental studies of dynamic stability were discussed. An experiment was suggested which would test the theoretical prediction that experimental results for nonlinear damping coefficients, originally dependent on a particular method of measurement, could be rendered generally applicable.

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